Spatially Contiguous Land Management: a sealed bid auction format

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Abstract

Ecosystem services are deteriorating. It is essential to develop economic instruments that promote the production of ecosystem services. Conservation agencies use, among other things, payment systems for ecosystem services that remunerate private landowners to adopt pro-environmental practices on their spatially contiguous lands. Iterative or sealed bidding procedures are well suited to provide efficient incentive systems. Experiments have shown the superiority of iterative auctions. In order to better understand the processes implemented, we propose here to analyze the strategies of the landowners in the case of sealed bids auction format.

Keywords: Conservation auction ; ecosystem services ; coordination

JEL Codes: D44, Q15, Q24, Q57

1 Introduction

Ecosystem services are deteriorating, developing adapted economic instruments is a new challenge at the global level. In the majority of cases, threatened ecosystems are located on private land (Rolfe et al., 2009). It is therefore essential to look for measures that promote the production of ecosystem services by private landowners. Conservation agencies use, among other

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things, payment systems for ecosystem services that remunerate private landowners to adopt pro-environmental practices on their land. The production of ecosystem services very often requires coordination of spatially contiguous land management. As a result, auction procedures seem well suited to provide efficient incentive systems. Different procedures are possible ranging from sealed bidding auction format to iterative bidding auction format.

In experimental studies Rolfe et al. (2005) and Reeson et al. (2008) considered problems of spatial agglomeration. They evaluated bid performance for creating connected landscapes. The experiments of Rolfe et al. have been implemented with real landowners using both sealed bidding and iterative auctions. For iterative auction formats, landowners were acquiring information each round, while for the sealed sealed bid format, landowners may communicate. Experiments have shown that the spatial patterns obtained are less expensive in the case of iterative auctions. Based on a scoring rule, Banerjee et al. (2015) consider a Conservation Auction model. They study an iterative descending-price auction, that explicitly includes the spatial objective into the selection criterion in the presence of a limited fixed budget. More complex procedure was experimented by Banerjee at al. (2018).

We propose here to analyze in detail the strategies of the landowners in the case of a specific sealed bids auction. In order to clarify the impact of communication, we make the assumption in our model that landowner do not communicate with each other. We deduce behavior of the bidder strategies with respect to the landowner type.

2 The auction design

We consider a set of N landowners. Each landowner participates by submitting an amount of financial compensation to the auction. In exchange for this financial compensation, each owner undertakes to carry out preservation practices on his property.

We assume that each participant i has a private cost c_i and that this private value is drawn in an F distribution. Each participant i submits an amount b_i. x_i indicates the winning or losing character of the participant i: $x_i = 1$ if i wins, $x_i = 0$ if i loses. Each winning participant in the auction receives the amount of their bid.

From the point of view of the environmental agency, benefits are expected for each individual participant but also and especially when two contiguous participants adhere to the conservation process. We consider linear configuration. All except the participants located at the extremities of the stream, interior participants have two neighbors. This translates into the environmental value function:

$$V((x_i)_i) = \sum_{i=1}^{N} m_i x_i + 2w \sum_{i=1}^{N-1} x_i x_{i+1}$$

where m_i is the benefit parameter for individual i and $2w$ the joint benefit parameter for two neighbors.

The cost corresponding to the landowner submissions is given by: $\sum_{i=1}^{N} b_i x_i$. Two alternatives are available, the cost may be limited to a given level or integrated in the gain function:

$$G((x_i)_i, (b_i)_i) = \sum_{i=1}^{N} m_i x_i + 2w \sum_{i=1}^{N-1} x_i x_{i+1} - \sum_{i=1}^{N} b_i x_i$$

We consider the second case with $m_i = m$ for all i. We assume that distribution of private cost value F is a common knowledge of the participants but not by the environmental agency. This hypothesis is fundamental: the environmental agency cannot have a strategic behavior.

2.1 Bidder strategy with score function announced

2.1.1 Continuous cost value Distribution

If we assume that distribution F is continuously differentiable, based on the first-order optimality condition, in case of uniform distribution F it is possible to deduce a continuous strategy (see Appendix):

$$B_l(c) = \begin{cases} m & \text{if } c < c_* \\ B_0(c) & \text{if } c_* < c \leq \min(\phi(c_*), \overline{c}) \end{cases} \tag{1}$$

with B_0 and ϕ linear in c, but also discontinuous strategies parametrized by x_*:

$$B_{x_*}(c) = \begin{cases} b_1 & \text{if } c < x_* \\ b_2 & \text{if } c > x_* \end{cases} \tag{2}$$

An iterative procedure

To obtain the bidders' strategy, an alternative procedure based on the definition of Nash equilibrium can also be used. It consists in constructing successive approximations of the bidder strategy $B_1, .., B_n$ based on maximizing the expected gain of the participant $E[c_i, b|B_{n-1}(.)]$ assuming that the other participants use the bidder strategy B_{n-1}:

$$E[c_i, B_n(c_i)] = \max_b E[c_i, b|B_{n-1}(.)]$$

Using this numerical procedure, starting with $B_1 = B_l$, depending on the chosen estimation of the expectation, the method converges to one of the discrete solutions B_{x_*}. This confirms the existence of multiple equilibrium.

2.1.2 Discrete cost value Distribution

In order to remove the ambiguities relating to the multiplicity of equilibrium we consider the case of discrete distributions for which we hope to be able to study in a more precise way. Assume that the private cost take K different values in ascending order $c^1 < c^2 < .. < c^K$. The discrete density of the private cost is: $f(c) = \sum_{k=1}^{K} \theta_k \delta_{c^k}$ with $\sum_{k=1}^{K} \theta_k = 1$. For $K = 2$ we obtain:

Proposition 2.1 *Assuming a discrete density of the private cost for two participants ($N = 2$) with two different type values ($K = 2$) and the environmental agency maximizes a nonnegative gain $G(x_i, b_i)$ then:*

(i) $\underline{m + w < c^2}$:

(ia) if ($c^1 > m$ or $c^1 \leq m, c^2 < m + 2w$) and $(1 - \theta_1)c^1 + (1 + \theta_1)c^2 < 2(m + w)$ there are an infinity of Nash equilibrium.

(ib) In the reverse cases, if $c^1 \geq m$ and $(1 - \theta_1)c^1 + (1 + \theta_1)c^2 > 2(m + w)$ or if $c^1 < m$ and ($(1 - \theta_1)c^1 + (1 + \theta_1)c^2 > (1 + \theta_1)(m + w)$ or $c^2 > m + 2w$), participants of type 2 do not bid and in the first case or if $c^1 + \frac{\theta_1}{1 - \theta_1} w > m$ in the second case then $b^1 = m + w$. If $c^1 + \frac{\theta_1}{1 - \theta_1} w \leq m$ in the second case, $b^1 = m$.

(ii) $\underline{c^2 \leq m + w}$:

if $(1 + \theta_1)c^2 - \theta_1 c^1 \leq m + w$ then the bidder strategies are given by:

$$b^1 = b^2 = m + w.$$

Proof: The gain net of costs to be maximize by the environmental agency is:

$$\max_{x_1, x_2 \in \{0,1\}} G(x_1, x_2, b_1, b_2) = m(x_1 + x_2) + 2w x_1 x_2 - (b_1 x_1 + b_2 x_2)$$

Each agent maximizes his expected gain:

$$E(c_i, b) = (b - c_i)Pr[x_i = 1]$$

We consider a Nash equilibrium, each participant follows the bid strategy $B(.)$ hence: $\max_b E(c_i, b) = E(c_i, B(c_i))$. Denote the optimal submission b^1, b^2 for the participant types. We successively consider the two bidder types c^1 and c^2.

(i) If $c^2 > m + w$, then $b^2 \geq c^2$, bidder of type 2 cannot win alone, hence his bid b must satisfy $b \leq 2m + 2w - b^1$, his expected gain is increasing with b, hence $b^2 = 2m + 2w - b^1$, so $b^1 < m + w < b^2$.

For bidder with lower value c^1, he is sure to win with $b \leq b^1$ and has a probability of winning θ_1 with $b^1 < b \leq b^2$, hence he respectively may bid b^1 and b^2, hence for a Nash equilibrium:

$$E_1(c^1, b^1) = b^1 - c^1 \geq \theta_1(b^2 - c^1) = E_1(c^1, b^2)$$

Hence at optimum, b^1 will be the highest value such that: $b^1 - c^1 \geq \theta_1(b^2 - c^1), b^1 + b^2 = 2m + 2w$ then $b^1 = 2m + 2w - b^2$ and we deduce $c^1(1 - \theta_1) + b^2(1 + \theta_1) \leq 2m + 2w$ and:

$$c^2 \leq b^2 \leq b^2_* = \frac{2}{1 + \theta_1}(m + w) - \frac{1 - \theta_1}{1 + \theta_1}c^1$$

Reversely optimal b^2 will be such that:

$$E_2(c^2, b^2) = \theta_1(b^2 - c^2) \geq E_2(c^2, b^1) = b^1 - c^2 (< 0)$$

hence positive gain implies $b^2 \geq c^2$, so $c^2 \leq b^2 \leq \overline{b}^2 = \frac{2}{1 + \theta_1}(m + w) - \frac{1 - \theta_1}{1 + \theta_1}c^1$.

If $c^1 < m$, participant of type 1 always wins with bid m, then we deduce that b^1 must not be lower than m, hence in this case: $\underline{b}^2 = c^2 \leq b^2 \leq \overline{b}^2 = \min(\frac{2}{1 + \theta_1}(m + w) - \frac{1 - \theta_1}{1 + \theta_1}c^1, m + 2w)$, and in the two cases:

$\underline{b}^1 = 2m + 2w - \overline{b}^2 \leq b^1 \leq \overline{b}^1 = 2m + 2w - c^2$. Hence we get an infinity of equilibrium.

In the reverse cases, bidding with participants of the two types is impossible, only participants of the first type can bid. If $c^1 > m$, $b^1 = m + w$. If $c^1 < m$, bidder of type 1 may bid m or $m + w$, his expected gain is respectively $m - c^1$ and $\theta_1(m + w - c^1)$, hence he bids m if $m - c^1 > \theta_1(m + w - c^1)$ i.e. if $c^1 + \frac{\theta_1}{1-\theta_1} w < m$ and bids $m + w$ if not.

(ii) If $c_2 \leq m + w$, two cases are available $2b^1 = 2b^2 = 2m + 2w$ or $b^1 + b^2 = 2m + 2w$ with $b^1 < b^2$. In the first case, the two participants always win, the gain are respectively: $m + w - c^1, m + w - c^2$. Considering the second case multiple strategies are available but we successively deduce:

$$
\begin{aligned}
\theta_1(b^2 - c^1) &\leq b^1 - c^1 = 2m + 2w - b^2 - c^1 \\
\theta_1(b^2 - c^2) &\leq 2m + 2w - b^2 - (1 - \theta_1)c^1 - \theta_1 c^2 \\
(1 + \theta_1)(b^2 - c^2) &\leq 2m + 2w - (1 - \theta_1)c^1 - (1 + \theta_1)c^2 \\
\theta_1(b^2 - c^2) &\leq \frac{2\theta_1}{1+\theta_1}(m + w) - \theta_1 \frac{1-\theta_1}{1+\theta_1} c^1 - \theta_1 c^2
\end{aligned}
$$

Hence the gain difference between the second and the first case satisfies:
$\theta_1(b^2 - c^2) - (m + w - c^2) \leq -\frac{1-\theta_1}{1+\theta_1}(m + w + \theta_1 c^1 - (1 + \theta_1)c^2) < 0$
hence gain is larger in the first case for the two types of participants. □

The proposition highlights the difficulty in obtaining bidder strategies. Participant gain can be lower for participant of higher type.

Discrete mixed strategies

In the case $c^2 > m + w$, we consider discrete and equidistant mixed strategies $\underline{b}^i + \frac{2j-1}{2n}(\overline{b}^i - \underline{b}^i), j = 1, .., n$ with respective probability p^i_j for participant of type $i = 1, 2$

with $n \geq 2$. We get the matrix of gain:

b^2 / b^1	$c^2 + \frac{\Delta b}{2n}$	$c^2 + 3\frac{\Delta b}{2n}$	.	$c^2 + \frac{2n-1}{2n}\Delta b$
$\underline{b}^1 + \frac{\Delta b}{2n}$	$(\underline{b}^1 - c^1 + \frac{\Delta b}{2n}, \theta_1 \frac{\Delta b}{2n})$	$(\underline{b}^1 - c^1 + \frac{\Delta b}{2n}, 3\theta_1 \frac{\Delta b}{2n})$	.	$(\underline{b}^1 - c^1 + \frac{\Delta b}{2n}, \frac{2n-1}{2n}\theta_1 \Delta b)$
$\underline{b}^1 + 3\frac{\Delta b}{2n}$	$(\underline{b}^1 - c^1 + 3\frac{\Delta b}{2n}, \theta_1 \frac{\Delta b}{2n})$	$(\underline{b}^1 - c^1 + 3\frac{\Delta b}{2n}, 3\theta_1 \frac{\Delta b}{2n})$	.	$(\theta_1(\underline{b}^1 - c^1 + 3\frac{\Delta b}{2n}), 0)$
.	.	.	.	.
$\underline{b}^1 + \frac{2n-3}{2n}\Delta b$	$(\underline{b}^1 - c^1 + \frac{2n-3}{2n}\Delta b, \theta_1 \frac{\Delta b}{2n})$	$(\underline{b}^1 - c^1 + \frac{2n-3}{2n}\Delta b, 3\theta_1 \frac{\Delta b}{2n})$	.	$(\theta_1(\underline{b}^1 - c^1 + \frac{2n-3}{2n}\Delta b), 0)$
$\underline{b}^1 + \frac{2n-1}{2n}\Delta b$	$(\underline{b}^1 - c^1 + \frac{2n-1}{2n}\Delta b, \theta_1 \frac{\Delta b}{2n})$	$(\theta_1(\underline{b}^1 - c^1 + \frac{2n-1}{2n}\Delta b), 0)$	.	$(\theta_1(\underline{b}^1 - c^1 + \frac{2n-1}{2n}\Delta b), 0)$

where $\Delta b = \overline{b}^1 - \underline{b}^1 = \overline{b}^2 - \underline{b}^2$. Probabilities are deduced from the equality of earnings for the different discrete strategies. Numerical results highlight that significant probability values are concentrated for b^1_n and b^2_1 with the following rankings: $p^1_1 > p^1_2$, $p^1_2 < .. < p^1_n$ and $p^2_1 > .. > p^2_n$. Moreover, once again, expected gain for participant of type 2 ($\theta_1 \frac{\Delta b}{2n}$) is lower than for participant of type 1 ($\underline{b}^1 - c^1 + \frac{\Delta b}{2n}$).

In many cases, sum of bid's participants are equal to $2(m + w)$ generating in case of a high value for the agency, a null gain for agency based on announced scoring function. It is therefore relevant for the agency not to use its real values and to announce in its score criterion values lower than these values.
Communication is not necessary for coordination of participants.

2.2 The score function is not completely announced

We now consider that w is not announced as in Elyakime et al. 1994., hence not perfectly known by the participants. We assume that $w = w_1 + \rho(w_2 - w_1)$ and ρ is drawn from the cumulative distribution G with support $[0, 1]$ We consider discrete cost value distribution with

$c^2 > m + w_2$. In this case, the probability that the principal gain is positive is equal to: $Pr[b^1+b^2 \leq 2m+2w_1+2\rho(w_2-w_1)] = Pr[\rho \geq \frac{b^1+b^2-2m-2w_1}{2(w_2-w_1)}] = 1-G(\frac{b^1+b^2-2m-2w_1}{2(w_2-w_1)})$ with $2m+2w_1 \leq b^1+b^2 \leq 2m+2w_2$, the expected gain for participant of type 1 is given by:

$$\begin{aligned}E_1(c^1,b^1) =&(b^1-c^1)(\theta_1+(1-\theta_1)(1-G(\frac{b^1+b^2-2m-2w_1}{2(w_2-w_1)}))\\ \geq&\theta_1(b^2-c^1)(1-G(\frac{b^1+b^2-2m-2w_1}{2(w_2-w_1)}) = E_1(c^1,b^2)\end{aligned}$$

Hence at optimum, b^1 will be the highest value such that: $E_1(c^1,b^1) \geq E_1(c^1,b^2)$, the expected gain for participant of type 2 is given by:

$$E_2(b^2,c^2) = \theta_1(b^2-c^2)(1-G(\frac{b^1+b^2-2m-2w_1}{2(w_2-w_1)})) \geq b^1-c^2$$

From $b^1 < c^2$ the inequality is always valid. The maximization with respect to b^2 gives:

$$2(w_2-w_1)(1-G(\frac{b^1+b^2-2m-2w_1}{2(w_2-w_1)})) = (b^2-c^2)G'(\frac{b^1+b^2-2m-2w_1}{2(w_2-w_1)})$$

Hence strategy of the participant depends on the distribution of the parameter w for the environmental agency. Assuming a uniform distribution:

$$\begin{aligned}b^1 =&m+\frac{w_2-\theta_1 w_1}{1-\theta_1}+\frac{c^1-b^2}{2}\\ b^2 =&m+w_2+\frac{c^2-b^1}{2}\end{aligned}$$

We deduce:

$$\begin{aligned}3b^1 =&2m+\frac{2}{1-\theta_1}((1+\theta_1)w_2-2\theta_1 w_1)+2c^1-c^2\\ 3b^2 =&2m+\frac{2}{1-\theta_1}((1-2\theta_1)w_2+\theta_1 w_1)+2c^2-c^1\end{aligned}$$

with the conditions relative to $b^1 + b^2$ and $b^i \geq c^i$:

$$2m + \frac{2}{1-\theta_1}((3-2\theta_1)w_1 - (2-\theta_1)w_2) \leq c^1 + c^2 \leq 2m + \frac{2}{1-\theta_1}((1-2\theta_1)w_2 + \theta_1 w_1)$$

3 Discussion and conclusion

We show that it is possible to obtain the bidder strategies. We find that the landowner with the largest cost may have a lower gain.

For $N > 2$, bidder strategies are attainable by means of further calculations. It would be possible to test the confirmation of the previous result in a more general framework. In conclusion, we showed that it is possible to obtain the bidder strategies.

We assume that distribution of private value F is a common knowledge of the participants but not by the environmental agency, hence the environmental agency cannot have a strategic behavior. If the environmental agency knows the distribution F, her can have a strategy: announce a lower parameters to build the scoring criterion or do not announce parameters of the score function.

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Appendix

We assume that the principal has a non negative gain. Each agent has a private value with distribution F of finite support and maximizes his expected gain:

$$E(c_1, b) = (b - c_1)Pr[x_1 = 1]$$

We consider a Nash equilibrium, each agent follows the bid strategy $B(.)$ hence: $\max_b E(c_1, b) = E(c_1, B(c_1))$

Let $N = 2$, the program of the principal is to maximize his gain:

$$\max(m - b_1, m - b_2, 2m + 2w - b_1 - b_2)$$

If $b_1 > m + 2w$ then $2m + 2w - b_1 - b_2 < m - b_2$ so $x_1 = 0$. If $b_1 < m + 2w$ then $m - b_2 < 2m + 2w - b_1 - b_2$ so the maximum reached for the principal is $\max(m - b_1, 2m + 2w - b_1 - b_2)$. We deduce that if the principal wants to insure a gain greater than G_0, the bid b of the agent is given by the maximum:

$$\max((m - c_1)I_{m-c_1 \geq 0}, (b - c_1)Pr[2m + 2w - b - B(u_2) \geq 0])$$

with $Pr[2m + 2w - b - B(u_2) \geq 0] = F(B^{-1}(2m + 2w - b))$. We first consider the problem of the right program which maximizes $(b - c_1)Pr[2m + 2w - b - B_0(u_2) \geq 0])$ with bid strategy B_0. The corresponding first-order condition is given by:

$$F(B_0^{-1}(2m + 2w - B_0(c))) = (B_0(c) - c)\frac{F'}{B_0'}(B_0^{-1}(2m + 2w - B_0(c)))$$

Let the function ϕ defined by: $B_0(\phi(c)) = 2m + 2w - B_0(c)$, we deduce:

$$B_0'(\phi(c))F(\phi(c)) = (B_0(c) - c)F'(\phi(c)) \quad (3)$$

$$B_0(c) + B_0(\phi(c)) = 2m + 2w \quad (4)$$

Assume F uniform distribution, $F(x) = x$ then, the system has a linear solution: $B_0(c) = \frac{2}{3}(m + w) + \frac{c}{2}$ with $\phi(c) = \frac{4}{3}(m + w) - c$. Due to limited value for bid strategy we deduce that the maximal private value which permits the agent to bid satisfies: $\frac{1}{3}(2m + 2w) + \frac{v}{2} \leq m + 2w$

i.e. $c \leq \overline{c}$ with $\overline{c} = \dfrac{2m+8w}{3}$. Let c_* the private value at which the agent changes of strategy:

$$m - c_* = (B_0(c_*) - c_*)\phi(c_*) = (\frac{2}{3}(m+w) - \frac{c_*}{2})(\frac{4}{3}(m+w) - c_*)$$
$$2(m - c_*) = (\frac{4}{3}(m+w) - c_*)^2$$

Hence $B(c) = m$ for $c < c_*$. Moreover $B_0(\phi(c_*)) = 2m + 2w - B_0(c_*) = \phi(c_*)$. As $B_0'(c) = \dfrac{1}{2}$, so it is impossible that $B_0(c) \geq c$ for $c > \phi(c_*)$. Hence:

$$B_l(c) = \begin{cases} m & \text{if } c < c_* \\ B_0(c) & \text{if } c_* < c \leq \min(\phi(c_*), \overline{c}) \end{cases} \tag{5}$$

But we have not shown the uniqueness of the solution. So other strategies are available, notably strategies provided by step functions. And this is indeed the case, if we consider a B function defined by:

$$B(c) = \begin{cases} b_1 & \text{if } c < x_* \\ b_2 & \text{if } c > x_* \end{cases} \tag{6}$$

with $b_1 \geq x_*$.

The optimality conditions are the following:

$$\begin{cases} b_1 - c \geq x_*(b_2 - c) & \text{if } c < x_* \\ x_*(b_2 - c) \geq b_1 - c & \text{if } c \geq x_* \end{cases} \tag{7}$$

with $b_1 + b_2 = 2m + 2w$. The inequalities are checked if and only if:

$$b_1 - x_* \geq x_*(b_2 - x_*) \tag{8}$$

$$x_*(b_2 - x_*) \geq b_1 - x_* \tag{9}$$

Hence $b_1 = b_2 x_* + x_*(1 - x_*)$, from $b_1 + b_2 = 2m + 2w$ we deduce $(b_1 - x_*)(1 - x_*) = 2(m + w)x_*$ with $b_1 \geq x_*$ and $0 < x_* < 1$. So it exists an infinity of function B satisfying the optimality conditions. Moreover, if we consider a B function defined by:

$$B(c) = \begin{cases} b_1 & \text{if } c < x_*^1 \\ . & \\ b_i & \text{if } x_*^{i-1} < c < x_*^i \\ . & \\ b_n & \text{if } c > x_*^{n-1} \end{cases} \tag{10}$$

with $b_i \geq x_*^i, i = 1..n - 1$. By similar reasoning we obtain that:

$$b_1 - x_*^1 = x_*^{n-1}(b_2 - x_*^1) \tag{11}$$

$$.$$

$$x_*^{n-i+1}(b_i - x_*^i) = x_*^{n-i}(b_{i+1} - x_*^i) \tag{12}$$

$$.$$

$$x_*^1(b_n - x_*^n) = x_*^2(b_{n-1} - x_*^n) \tag{13}$$

with $b_i + b_{n-i} = 2(m + w)$ for $i = 1..E[\frac{n}{2}]$ and the existence of infinity of function B.