On Artesian Aquifers : Pressure as a New Common

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Abstract

VERY PRELIMINARY VERSION

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1. Introduction

Common-pool resources (CPRs) have been extensively studied in the literature since the seminal paper of Gordon [12], or the Tragedy of Commons of Hardin [13]. Most of these papers address issues related to the characteristics of non-excluability and rivalry in consumption. Indeed it is difficult to assign adequate property rights to control access to resource stock, and any amount of resources which is extracted is not available for others anymore. However, the degree of rivalry can differ, as well as the resulting externalities. Private appropriation especially reduces the available stock, generating a series of externalities associated with stock variation. Such *stock externalities* e.g. occur when harvesting almost all common pool resources. This is, of course, not the case for resources where the stock is infinite (e.g. solar energy resource). But it may also not true for some resources available in finite amount. This is especially the case for *confined aquifers*, i.e. aquifers confined between an upper and a lower impermeable layer which obtain their recharge from a distant and more elevated aquifer which is often unconfined. For these aquifers, when drilling a well, water naturally flows out without any pumping (i.e. the *artesian property*) and the withdrawal from the confined aquifer is immediately compensated so that there is no dewatering at all. The absence of stock externality therefore suggests that this resource does not suffer overexploitation under open access, and

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needs no regulation. The main objective of this paper is to show that this intuition is wrong. The basic point beyond this paper relay on the idea that the artesian property of an individual well is mainly related to the global water pressure inside a confined aquifer. This *pressure externality* leads under an open access regime to an excessive number of wells and an overexploitation of the resource. The result is obtained by contrasting the outcome with open access to the one induced by a socially-optimal well drilling strategy which internalizes this externality. In other words, even if there is no decrease in groundwater stock, we show that the behaviors of economic agents are not aligned with the socially-optimal outcome.

Most of the litterature on groundwater mamagement never considers this case. Since Smith [32] seminal contribution, stock externalities reflects the effects a *reduction* of a resource stock may have on economics decision. A lower stock actually creates two distinguished effects (Burt and Provencher [5]): A reduction in extraction opportunities to all others, and an increase in future extraction costs for all users. Stock externality actually represents the former effect, and arises because exploitation of resource is constrained by a finite resource stock. The latter effect represents the pumping cost externality, and arises because the cost of extraction depends on the resource stock. Most of the studies however irrespectively used to refer to one or the other, but only include pumping cost externality in their model (e.g. Gisser and Sanchez [11]).

Groundwater has been extensively studied since the seminal paper of Gisser and Sanchez [11], who conclude that welfare gains from public management are negligible. A large part of the follow-up literature still analyzes the potential role of water management under different assumptions, but no clear-cut answer has been provided. A number of studies compares perfect competition with socially-optimal management outcomes. For instance, Provencher [25], and Provencher and Burt [26] respectively show that property rights allow to recover a large part of welfare gains, but these gains remain relatively low. Allen and Gisser [1] still get a small difference between competition and social planner, while Brill and Burness [3] find an increased divergence between both scenarios under different hydrologic and economic assumption including demand growth, declining well yields or low social discount rates. Another part of the literature rather contrasts the social planner'solution with strategic behaviors (Negri [22], Rubio and Casino [28]). Rubio and Casino [28] for instance confirms Gisser and Sanchez result. Some papers actually expand this literature in a number of directions, including uncertainty (Knapp and Olson [16], Tsur and Graham-Tomasi [39]), or conjunctive surface water use (Azaiez [2], Pongkijvorasin and Roumasset [24], Stahn and Tomini [34] and [35]). All this work nevertheless assumes that aquifer behaves as a "bathtub" with perfect hydraulic conductivity. This assumption is equivalent to assume a bottomless aquifer, and enable to capture only one effect resulting from stock variation, that is the pumping cost externality. Recent papers introduce a spatial representation of the aquifer considering that transmissivity is not infinite, such that the stock available to users, and the impact on users' decisions may differ according to the location of extraction (Brozovic et al. [4], Chakravorty and Roumasset [6], or Chakravorty and Umetsu [7]). Brozovic et al. [4] establish that welfare gains may be mis-estimated under the "bathtub" assumption. As a consequence, those studies show that policy recommendations based on such basic hydrologic assumptions may be irrelevant. Nevertheless, all these studies analyze the magnitude of externalities and inefficiency within the specific context of unconfined aquifers.

To the best of our knowledge, only Worthington et al. [40] consider confined aquifers. Adopting a dynamic programming system applied to the Crow Creek Valley aquifer, they characterize the optimal seasonal water and contrast it with the competitive outcome. They show that there may be significant welfare gains from a public management. They however fail to capture specific features of confined aquifers, since they merely expand the standard model based on a simple balance equation of water stock, considering effect of pumping activities on groundwater stock. Artesian aquifers are groundwater reservoirs confined by impermeable layers where water is under pressure, such that water may naturally flow over the top of a drilling well, without the need of pumping. But the returns of each well is typically related to the pressure of the confined aquifer. Drilling a new well reduces the pressure of the whole aquifer and therefore exerts a negative externality on the returns of existing wells. This is precisely what we have called a pressure externality which requires an overall management of the number of wells.

There exist many artesian aquifers all over the world. The Great Artesian basin in Australia is considered as the largest and deepest artesian basin in the world, underlying 22% of the continent. Other important systems are the Edwards aquifer in Texas (USA), or the Northern Saharan Aquifer System in the north of Africa. We also find artesian wells in the South of the city of Vancouver, or in several areas in India which provide water for millions people. Nevertheless, this resource is subject to the “rule of capture”, and thus threatened by human pressure. Moreover, the flow of many wells remains uncontrolled. As such, optimizing the exploitation of this resource, and the production of groundwater reservoirs, is a strong challenge. Some regulations have already been implemented in some areas to limit adverse impacts. For instance, the St John River Water Management District encourages wells owners to control flow, and even abandon problem wells by properly plugging them.

In this paper, we develop an hydro-economic dynamic model based on fluid mechanism to adequately consider time evolution of water pressure in the aquifer. We specifically use the fluid dynamics to describe the relationship between water stock, pressure, and water discharge. On this basis, we first assume that this resources is under a open access regime. This means that additional wells are drilled until the marginal cost of an additional well is equal to the private marginal returns, this last quantity being, under a pure competitive assumption, in relation with the water price. In a second step, we introduce an optimal management problem of the number of wells which takes into account this pressure externality. we characterize, for both management regimes, the long-run steady values and there property with respect to changes in the economic and hydrologic parameters of the model. We finally compare these two management regimes. For that purpose we consider the open access case and introduce two rates. The first measures the yield losses per wells compared to optimal management case while the second measures the rate of change on the long run aquifer level. These rates stands respectively for proxies of the loss of pressure and the overextraction behavior. we also provide a sensitivity analysis of these rates to the economic and hydrologic parameters of the model.

The remainder of the paper is organized as follows. The specific features of a confined aquifer especially concerning the consequences of the water pressure model are presented in Section 2. Section **??** presents the main economic characteristics of our system and analyses the short and long run properties of this hydro-economic model under free access. Section 4 analyses the centralized water management problem Section **??** contrasts this two regimes in term of pressure losses and overextraction, and Section **??** concludes. Proofs are relegated to an appendix.

2. Aquifer systems with flowing artesian wells

Confined aquifer is a relevant example of common-pool resource for which rivalry may be nearly questioning. Indeed, units of water withdrawn from such aquifers are not available anymore, but the same amount of the total stock in the ground is still available for others. In other words, there may be private appropriation of water, but the availability of the resource is not reduced. What is nevertheless reduced is water pressure. More precisely, confined (artesian) aquifers are overlain by a relatively impermeable layer, such that water is under a pressure greater than atmospheric. Consequently water may rise above the top of the aquifer, even above the land surface when a drill hole penetrates the aquifer. We especially observe wells, which naturally flow to (or above) the land surface, without the need of pumping, when the *potentiometric line*,[1], is above the surface. This level furthermore represents the pressure exerted by water, given the force of gravity. Consequently, the confined layer remains saturated, even with exploitation. However, abstraction of groundwater resources implies variations in pressure, which in turn trigger falls in water flowing at ground surface.

These different characteristics result from a specific geological structure of confined aquifer. We used to distinguish two parts: the *outcrop* area, or the recharge zone, and the confining area. The first part is usually located at a high elevation (e.g. near mountains) exposed at the ground surface, at a considerable distance away from the second part which is at a lower elevation, beneath an impermeable layer. Broadly speaking, the first part behaves as a "bathtub" unconfined aquifer with a flat bottom of area A_r and perpendicular sides, where we observe a water table level $h \in (0, h_{\max})$, which may rise and fall according to recharge and artificial discharge. This specific level corresponds here to the potentiometric line. The second part is of area A_c, and is already full to its capacity. This part is referred hereafter as the artesian aquifer. We moreover assume that the elevation of the artesian aquifer is normalized to zero, such that the elevation of the water table defines the difference between the piezometric line and the elevation of

[1] The potentiometric line is an imaginary line where water pressure is equal to atmospheric pressure. Water will rise to this specific level. In an unconfined aquifer, the potentiometric line is equivalent to the water table level.

the upper surface of the confined area. These assumptions ensure the existence of flowing artesian wells.

Let us notice R the *potential* recharge, i.e. all water available at the surface, which may or may not reach the ground. The part of the potential recharge that effectively soaks into the soil represents the *actual* recharge, while the remaining water runs off over land.[2] In other words, only the proportion $(1 - \rho)R$ recharges the aquifer. We finally assume that there is no exogenous discharge. The figure 1 illustrates this specific structure.

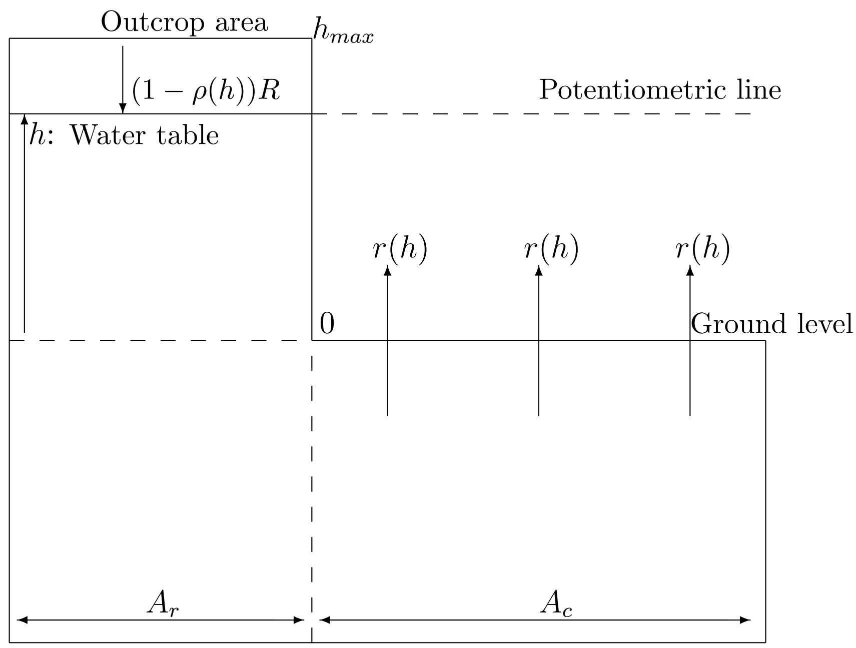

Figure 1: Schematic representation of a confined aquifer

The hydrodynamics of confined aquifers results from flows and interaction between those two parts. More precisely, the actual recharge enters the upper outcrop area and moves slowly down toward the confining layer. When there exists a well, water may flow out of this well, because of the pressure exerted by the weight of groundwater, without any change in the volume of water within the confining part. Artesian groundwater exploitation actually results in changes in volume storage in the outcrop area only, and consequently in pressure declines. Since there is less pressure to cause a well to naturally flow to the land surface, less water will rise to the surface. From fluid dynamics, we can characterize the relationship between water pressure and the water table level, h, and

[2] This distinction is commonly used in the hydrogeological literature (e.g. Rushton [30]).

then deduce the *maximum water yield* flowing out a well using the Torricelli's formula.[3] More precisely, let us assume that k is the diameter of the drill hole, i.e. a technological characteristics, and g is the acceleration due to gravity, the maximum water yield, r, is approximately given by the following equation:[4]

$$r(h) = k\sqrt{2gh} \tag{1}$$

We can easily observe that the lower water table, the lower the maximum water yield ($r'(h) > 0$). This means that a lower pressure is exerted by water, and thus a smaller amount of water flows at the surface. By abuse of language, water yield approximatively captures the pressure (given the technology k), and changes in water yield is equivalent to changes in artesian pressure. Moreover, if there are n active wells operating at capacity, the total water discharge is of $w(h) = r(h)n$. We will consider n, in the rest of the paper, as a real number. This means, for instance, that the last well is only partially active.

Hence, we can characterize the hydrodynamics of confined aquifers by two equivalent formulations: a law motion based on changes in the water table levels, $h(t)$, in the outcrop area, or a law motion based on changes in the maximum water yield, $r(t)$, resulting from changes in the pressure of water in the confining part. The dynamics of water table levels results from a physical water balance between inflows and outflows: the actual recharge $(1-\rho)R$ is the source of incoming water, and outflows are the sum of water discharge flowing at the surface in each well, $w(h(t)) = r(h(t))n(t)$. Within this context, time evolution of the water table (as long as $h \geq 0$) in the outcropping area is associated to changes in the piezometric water head of the whole aquifer, and is described as follows:

$$\dot{h}(t) = \frac{(1-\rho)\,R - w\,(h(t))}{sA_r} \tag{2}$$

with s the storativity coefficient of the aquifer. We can notice that this dynamics is state-dependent on the contrary to the law motion of water table in unconfined aquifers.

Using Torricelli's formula (1), we can characterize the second dynamics. Indeed, any additional well allow additional outflows, but this also mean that water pressure will be reduced, and consequently water yields of existing wells. From equation (1), we get the time evolution the maximum water yield:

$$\dot{r}(t) = \frac{kg\dot{h}(t)}{\sqrt{2gh(t)}} = \frac{k^2 g\dot{h}(t)}{r(t)} \tag{3}$$

Using the dynamics (2), it follows that the dynamics of the yield of an artesian well is given by:

$$\dot{r}(t) = \frac{k^2 g}{r(t)sA_r}\left[(1-\rho)\,R - r(t)n(t)\right] \tag{4}$$

[3] This theorem is an application of the Bernouilli's theorem relating the pressure, velocity and elevation for a steady flow system. The Toricelli's formula is a statement for fluid flowing out of an orifice.

[4] This approximation is based on the motion of a frictionless and incompressible fluid (i.e. Bernouilli's hydrodynamic formula) where the water velocity inside the aquifer and the discharge of the artesian well is assumed to be negligible (since diameter of the well is negligible with respect to the area of the aquifer)

This second dynamics sounds more economically-oriented since outflows results from the economic decision to build new wells, but it is quite unusual because it is based on the accumulation of resource flowing out a reservoir. However, more water out of the aquifer means that the pressure is correspondingly altered. This formulation interestingly captures that the pressure of the water behaves as a common-pool resource. Consequently, we may expect the existence of a new externality, different from pumping cost contrary commonly observed in analysis of unconfined aquifers. This *pressure externality* results from the effect of an additional well on future yields.

It remains to set the initial condition of this system. For simplicity, we assume for Eq. (2) that the initial level of the water table is $h(0) = h_{\max}$ or, equivalently, for Eq. (4) that the initial yield is $r(0) = k\sqrt{2gh_{\max}}$.

3. Confined aquifer as an open access resource

As recharge occurs at elevated areas, we intuitively assume that exploitation of such an aquifer occurs above the confined part of the aquifer in the distant lowest areas like plains or valleys where the economic activity take place. The water demand is described by an decreasing inverse water demand for the overall production of water, $P(w)$ with $P'(w) < 0$ and and providing a social benefit $\int_0^w P(\omega)d\omega$. Moreover, we ignore possible conveyance losses, and we thus assume that all water flowing at the surface is the actual volume of water used by consumers.

Individual wells are built to fullfill this demand, such that the number of wells correspond to the number of well owners. The well owner faces an instantaneous profit function:

$$\Pi = p(t)r(t) - c \tag{5}$$

with $p(t)$ the market price the owner receives for the yield $r(t) = k\sqrt{2gh(t)}$ of her well, and $c > 0$ the annual exploitation cost of her well. Following the literature on irrigation networks (e.g. Chakravorty and Roumasset [6]; or Jandoc, Juarez, and Roumasset [14]), this cost function includes among others the per-period equivalent of construction cost, operating and maintenance cost, and water provision because of irrigation networks, conveyance structures linking all that wells and distribution to consumers. Now consider that a confined aquifer is exploited by myopic well owners, who decide to exploit the aquifer when there is a positive profit. The resource being in open access, this also means that new wells are drilled as long as the observed profit remains positive. This free-entry condition therefore induces the following dymanics of the number of wells:

$$\dot{n}(t) = \alpha \left(P\left(n(t)r(t)\right) r(t) - c\right) \tag{6}$$

with $\alpha > 0$ an adjustment parameter.

Even if there is no dewatering of the confined par of the aquifer, the yield of any additional well however reduces the water table level, $h(t)$, in the outcrop area. But a lower water table in the outcrop area reduces the pressure in the confined aquifer and therefore lowers the yields of each well. A complete description of the dynamics of the number of well and their yields can be obtained either by combining the free-entry dynamics (Eq.(6) with those of the yields (Eq.(4) or, with the help of the Toricelli's formula linking the return to the water table, by combining the free-entry dynamics (Eq.(6) with those of the water table in the outcrop area (Eq. 2). Hence we have:

$$\begin{cases} \dot{n}(t) = \alpha\left(P\left(n(t)k\sqrt{2gh(t)}\right)k\sqrt{2gh(t)} - c\right) \\ sA_r\dot{h}(t) = (1-\rho)R - n(t)k\sqrt{2gh(t)} \end{cases} \tag{7}$$

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The open access hydro-economical equilibrium is thus obtained by setting the time variation of the number of wells, and that of the water table levels in the outcrop area equal to 0, $\dot{n} = \dot{h} = 0$. We easily deduce that the number of wells under open access is as follows:

$$n^m = \frac{(1-\rho)R}{c}P\left((1-\rho)R\right) \tag{8}$$

This number depends on the actual recharge, weighted by somehow a profitability rate.

We then derive the steady state value for the water table level:

$$h^m = \frac{1}{2g}\left(\frac{c}{kP\left((1-\rho)R\right)}\right)^2 \tag{9}$$

Then using the Toricceli formula (1), we can respectively derive the long run maximum water yield:

$$r(h^m) = \frac{c}{P\left((1-\rho)R\right)} \tag{10}$$

and the global water consumption:

$$w(h^m) = P\left((1-\rho)R\right) \tag{11}$$

We can even prove that the open access hydro-economic equilibrium is stable, as it summarizes in the following Proposition.

Proposition 1. *There exists a unique steady state which satisfies local saddle point stability.*

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Table 1: Comparative statics

Parameters	Effects on the water table	wells number	water consumption	water yield
Cost (c)	+	-	0	+
Diameter of the well (k)	-	0	0	0
Recharge (R)	+	$+/-$	+	+
Infiltration rate (ρ)	-	$+/-$	-	-

4. Optimal management of an artesian aquifer

The objective of the social planner is as usual to choose the highest level of groundwater exploitation, accounting for the hydrodynamics of the aquifer. She can thus use one of the two formulations to characterize the optimal supply of water to fulfill the overall water demand. First, assume that the social planner aims to control at each time period t the amount of water flowing out wells, she will choose the number of active wells, $(n(t))$, accounting for the exploitation cost, cn, and given the flowing water dynamics (4). If $\delta > 0$ denotes the discounting rate, she maximizes the discounted sum of social benefits net of the sum of the well exploitation costs:

$$\begin{aligned} &\max_{(n(\cdot))} \int_0^{+\infty} \left(\left(\int_0^{r(t)n(t)} P(\omega) d\omega \right) - cn(t) \right) e^{-\delta t} dt \\ &\text{s.t.} \quad \dot{r}(t) = \frac{k^2 g}{r(t) s A_r} \left[(1-\rho) R - r(t) \sum_{i=1}^{m} n_i(t) \right] \end{aligned} \tag{12}$$

If now the social planner wants to choose the water supply per area, $w(t)$, which is equivalent to the choice of the number of wells per area she will account for the usual dynamics of the water table (2). Remember that the instantaneous yield of each well is related to the water table (see Eq. (1). We can thus re-formulate the optimization programm following a more standard approach:

$$\begin{aligned} &\max_{(w(\cdot))} \int_0^{+\infty} \left(\left(\int_0^{w(t)} P(\omega) d\omega \right) - c \left(\frac{w(t)}{k\sqrt{2gh(t)}} \right) \right) e^{-\delta t} dt \\ &\text{s.t.} \quad sA_r \dot{h}(t) = (1-\rho) R - w(t) \end{aligned} \tag{13}$$

This quite apparent usual problem in the groundwater literature should however not be misinterpreted. First, the level $h(t)$ is not the water table of the aquifer, but the potentiometric level. Remember that this level corresponds to the water table in the outcropping area, while the confining part is never dewatered. Moreover, given our normalization elevation rule, the water table must be positive in order to ensure the artesian property. Secondly, the dependance of the cost function on the argument $h(t)$ cannot be used to interpret a pumping cost externality, even if this function is decreasing and convex in $h(t)$, since there are typically no pumping cost for artesian wells. The presence of $h(t)$

represents the pressure externality. For instance, a decrease of the piezometric water table reduces the water yield. This requires, for a given water production level, $w(t)$, to drill additional wells and therefore to globally increase the well management cost in a given area.

This new setting has however one failure. The co-state variable, $\lambda(t)$, associated with the water table dynamics in the program (13) stands as usually for the shadow price of the decrease of one unit in the water table. This variable only partially captures the shadow price of a unit of water naturally rising at the surface from an artesian well. This value is actually given by the co-state variable, $\Gamma(t)$, associated to the dynamics using in the optimization problem (12). This variable measures the monetary consequences of a decrease in the yield of an artesian well. However, let us respectively denote the future values at t of the two programs (12) and Eq. (13) along the optimal path by: $V_1^*(t) = V_1\left((n^*(t)), r^*(t), \Gamma^*(t)\right)$ and $V_2^*(t) = V_2\left((w^*(t)), h^*(t), \lambda^*(t)\right)$. We know from the equivalence of both programs that $\forall t$, $V_1^*(t) = V_2^*(t)$. It follows from Torricelli's formula (1) and the usual property of the co-states that:

$$\Gamma^*(t) = \frac{\partial V_1^*}{\partial r} = \frac{\partial V_2^*}{\partial h}\frac{\partial h}{\partial r} = \lambda^*(t)\left(\frac{r^*(t)}{k^2 g}\right) = \lambda^*(t)\left(\frac{\sqrt{2gh^*(t)}}{kg}\right) \tag{14}$$

In other words, we can get the shadow price of a unit of water naturally rising at the surface from an artesian well by using the solution the program given by Eq. (13).

The current value Hamiltonian becomes:

$$\mathcal{H}\left(w(t), h(t), \lambda(t)\right) = \int_0^{w(t)} P(\omega)d\omega - c\frac{w(t)}{k\sqrt{2gh(t)}} + \frac{\lambda(t)}{sA_r}\left[(1-\rho))\,R - w(t)\right] \tag{15}$$

where $\lambda(t)$ is the current-value shadow price associated with the water table. We derive the following first-order conditions:[5]

$$P(w) = \frac{c}{k\sqrt{2gh}}C' + \frac{\lambda}{sA_r} \tag{16}$$

$$\dot{\lambda} = \delta\lambda - \frac{cwg}{k\sqrt{(2gh)^3}} \tag{17}$$

Eq. (16) represents the usual optimality condition, which yields a marginal benefit in each period equal to the total marginal costs, the sum of the marginal exploitation cost and the water shadow price, $\frac{\lambda}{sA_r}$.[6]

Equation (17) describes the behavior of the shadow value. This equation shows that the time evolution of the shadow price depends on the discount factor, and as usually in the literature on unconfined aquifer, it depends on the marginal exploitation cost. Actually

[5] We remove the time argument for a simpler reading.

[6] Remember that λ measures the shadow price of one additional drop of water in the ground, and $\frac{\lambda}{sA_r}$ is consequently the shadow value of a marginal water table elevation in the unconfined part of the aquifer.

a stock-dependent recharge impacts the value of the resource, since current exploitation of the confined aquifer affects the water table in the outcrop area, which in turn affects pressure, and consequently future flowing.

The analysis of these conditions will allow us to characterize the optimal pathes, as it is summarized in Proposition 2:

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Proposition 2. *Any optimal water consumption path,* $\tilde{w}(t)$ *and water table path,* $\tilde{h}(t)$*, satisfy the optimal first conditions (16) and (17), with the dynamics (2).*

Moreover under the usual transversality condition, $\lim_{T\to\infty} e^{-\delta T}\lambda(T) = 0$*, the shadow price* $\lambda(t)$ *is positive and if we assume that the elasticity of the marginal exploitation cost is larger than one, the sufficient Mangasarian conditions are satisfied.*

Proposition 2 interestingly proves that the (current-value) shadow value of water head is positive at any period of time. Moreover, under a quite standard assumption on the elasticity of the marginal cost function, we prove that the optimal control (13) satisfies the sufficient Mangasarian conditions.

5. Sustainable artesian water exploitation

We can now investigate the sustainable management of artesian aquifer by characterizing the optimal steady state, i.e. by setting $\dot{h} = \dot{\lambda} = 0$. Indeed, in the steady state, we characterize the highest rate of exploitation without depleting the aquifer, or rather without reducing to zero water pressure in the long run. From equation (2), we can directly derive the steady state level of global water consumption:

$$W = (1 - \rho)\, R \tag{18}$$

This exactly corresponds to the actual recharge.

Using equation (17), we obtain the steady state for the shadow value of water:

$$\lambda(h^*) = \frac{cg\,(1-\rho)\,R}{\delta k\sqrt{(2gh^*)^3}} \tag{19}$$

Then, using the Toricceli formula (1), we can respectively derive the long run maximum water yield of a well:

$$r(h^*) = k\sqrt{2gh^*} \tag{20}$$

and the shadow value of the pressure:

$$\Gamma(h^*) = \frac{c\,(1-\rho)\,R}{2\delta g k^2 h^*} \tag{21}$$

All these steady state values depend on the long run level of water head in the first part of the aquifer, h^*. On the basis of previous observations and using equation (16), we get the condition required for the level of water table at the steady state:

$$P\left[(1-\rho) R\right] = \frac{c}{k\sqrt{2gh^*}}\left(1 + \frac{(1-\rho) R}{2\delta h^* s A_r}\right) \tag{22}$$

Hence, we can investigate the existence of a steady state based on this single condition, and examine the stability properties. All the results are introduced in the Proposition below.

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Proposition 3. *There exists a unique steady state which satisfies local saddle point stability.*

We can now analyze the long run impact of variations in parameters of the model on water yield, and water consumption. More precisely, we want to study the effect of hydrological parameters, $\{A_r, R, m\}$,[7] and the economic parameters, $\{k, \delta\}$. Moreover, since all variables depend on the water table equilibrium, we easily guess the existence of two effects: a *direct effect* from variations of parameters, and an *indirect effect*, because variations in one parameter will also affect the water table level, which will, in turn, impact the variable of interest. Consequently, we will first apply the implicit function theorem on equation (22) to deduce the impact of a change in a parameter $\theta = \{A_r, R, m, k, \delta\}$ on the long run level of the water table. Let us write equation (22) as follows:

$$\phi\left(h; k, A_r, R, m, \delta\right) \equiv P\left(\left(1-\rho(h)\right) R\right) - \frac{1}{k\sqrt{2gh}} C'\left(\tfrac{(1-\rho(h))R}{mk\sqrt{2gh}}\right)\left(1 + \tfrac{(1-\rho(h))R}{2h(\delta s A_r + \rho'(h) R)}\right) = 0 \tag{23}$$

such that, we have:

$$\frac{dh}{d\theta} = -\frac{\frac{\partial\phi}{\partial\theta}}{\frac{\partial\phi}{\partial h}}$$

We can prove that $\frac{\partial\phi}{\partial h} > 0$, therefore we get that:[8]

$$sign\left\{\frac{dh}{d\theta}\right\} = -sign\left\{\frac{\partial\phi}{\partial\theta}\right\}$$

We easily observe that all parameters, except the natural recharge R, affect the long run costs only. This specifically modifies the incentives to exploit artesian water by affecting

[7] We do not analyze the impact of the storativity parameter, s, since the effect of this parameter is similar to the effect of the associated parameter A_r.

[8] All computations are presented in appendix.

the full marginal cost, i.e. the sum of the marginal exploitation cost, and the marginal user cost. Typically, the two parameters associated with exploitation, that is $\{m, k\}$, affect both parts of the full marginal cost. First, we can notice that a higher number of exploitation areas, m, is equivalent in this modeling to lower water production per area in the long run, which consequently drive to decrease marginal cost of exploitation. However, for the optimality condition (22) to hold in the long run, the water table level must decrease. Indeed, a fall in water table means that the actual recharge, $(1-\rho(h))R$, increases, which in turn implies that the inverse water demand declines. A lower water table will also modifies marginal cost, but the overall impact leads to a decreasing water table level in the long run. Similarly, a larger diameter of the drill hole, k, is equivalent to a decrease in the number of wells in the long run, which decreases marginal exploitation cost. Consequently, a larger drill hole leads to a decline in the long-run water table level.

Second, the hydrogeological parameter, A_r, and the discount rate, δ, are the two parameters affecting the time variation of the shadow value of the resource (see Eq. (17), and thus they impact its long run value. Basically, a larger recharge area, A_r, gives a greater incitation to exploit water, because of a lower future value of the resource. Likewise, the higher the discount rate, the lower the present value of the future benefit. Both parameters therefore lead to a lower water table level in the long run.

Finally, an increased potential recharge, R, impacts simultaneously water demand, instantaneous exploitation cost, and the long-run social cost. Actually, an increase in parameter R allows more water to infiltrate the aquifer, which consequently decreases water price. Intuitively, at the steady state, the higher the inflows, the higher the outflows of the system. This thus drives to increase marginal costs. The overall effect on demand and costs decrease the incitation to exploit artesian groundwater, and thus leads to observe a higher water table in the long run.

Table 2 summarizes the result of the comparative statics.

Table 2: Impact on the long run water table

Parameters	Effect
Number of patches (m)	$-$
Diameter of the drill hole(k)	$-$
Recharge area (A_r)	$-$
Discount rate (δ)	$-$
Recharge (R)	$+$

Let us now analysis the effect of variations in a parameter, $\theta = \{m, k, \delta, A_r, R\}$, on the long run water consumption, $W(h^*)$, the long run water yield, $r(h^*)$, and the number of wells per area, $n(h^*)$. For all parameters, except the potential recharge, R, it is a matter of fact that there is no direct effect on water consumption, and the impact exclusively depends on the effect of variations in the water table level. Specifically, the impact only depends on marginal variations of the net recharge because of changes in the water table level. From Table 2, we know that the higher one of these parameters $\theta = \{m, k, \delta, A_r\}$, the lower the water table level, and consequently the higher the net recharge. At the steady state, this thus results in a higher water consumption. The effect of variations in the potential recharge, R, sounds more ambiguous since it results from the combination of a positive direct effect, and a negative indirect effect, which both alter the net recharge. Indeed, a higher potential recharge drives to a decrease in the infiltration rate, since the water table level is increased, but also contribute to increase the net recharge. However, we can observe that some of effects offset each other in terms of their overall effect on the water demand, and the marginal production costs, such that, at the end, a higher potential recharge positively affects water consumption.

We can make similar observations concerning the analysis of changes in the long run water yield. Once again, all parameters, except the diameter of the drill hole, k, have no direct effect, and variations result from the impact of these parameters on the water table level. More precisely, an increase (resp. a decrease) in the water table level contributes to an increase (resp. a fall) in the amount of water flowing out of a well. To this end, an increase in one of these parameters, $\theta = \{m, \delta, A_r\}$, leads to decrease water yield, while a higher potential recharge, R, boosts water yield at the surface. Finally, a single parameter has an *a priori* ambiguous impact: a larger drill hole results in a positive direct impact, and a negative indirect impact, because of an abatement of the water table level. However, both effects alter the cost structure, and the impact of these opposite variations offset such that we observe a positive overall effect. Consequently an increased diameter of the drill hole drives to a higher long run water yield.

The analysis of impacts on the number of wells per areas is a little more tricky. Remember that $n(h^*) = \frac{W(h^*)}{mr(h^*)}$. Variations in the number of wells will thus depend on simultaneous changes in water consumption, and in water yield, and, for the single parameter m, it will also depend on a direct effect. For the two parameters $\theta = \{\delta, A_r\}$, it is quite straightforward, since we observe an increase in water consumption along with a decline in water yield in the long run. This thus results in a higher number of wells per area in order to increase water supply to satisfy water demand. Changes in the potential recharge, R, or in the diameter of the drill hole, k, leads to observe a same mechanism: both parameters positively affect water consumption, and water yield, but the overall effect is positive. This thus means that the impact on water consumption is higher than the impact on water yield. Finally, we observe a lower number of wells per area with respect to a higher number of exploitation areas, m. This means that the contribution of the direct effect of an increase in the parameter m is higher than the impact of changes in water consumption and water yield. Table 3 summarizes the results:

6. The nature of the overextraction

XXX

7. Concluding remarks

XXX

Table 3: Variations of water consumption, water yield, and the number of wells

Parameters	Impact on water consumption (W)	Impact on water yield (r)	Impact on the number of wells (n)
Number of patches (m)	+	-	-
Diameter of the drill hole (k)	+	+	+
Recharge area (A_r)	+	-	+
Discount rate (δ)	+	-	+
Recharge (R)	+	+	+

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A. Proof of proposition XXX

The dynamics is given by: (dans le texte)

$$\begin{cases} \dot{n}(t) = \alpha \left(P \left(n(t)k\sqrt{2gh(t)} \right) k\sqrt{2gh(t)} - c \right) \\ \dot{h}(t) = \frac{1}{sA_r} \left((1-\rho) R - n(t)k\sqrt{2gh(t)} \right) \end{cases} \tag{24}$$

(i) <u>*Local stability*</u>

From Eq.(24), observe that the linear approximation of these dynamics arround the steady state is given by:

$$\begin{pmatrix} \dot{n} \\ \dot{h} \end{pmatrix} = \underbrace{\begin{bmatrix} \alpha P'(w^m) k^2 2gh^m & \alpha \left(P'(w^m) n^m k^2 g + P(w^m) k\sqrt{g} (2h^m)^{-1/2} \right) \\ -\frac{k\sqrt{2gh^m}}{sA_r} & -\frac{1}{sA_r} \left(n^m k\sqrt{g} (2h^m)^{-1/2} \right) \end{bmatrix}}_{=D_m} \begin{pmatrix} \dot{n} - n^m \\ \dot{h} - h^m \end{pmatrix} \tag{25}$$

Under our assumption, it is immediate that:

$$trace(D_m) = \alpha P'(w^m) k^2 2gh^m - \frac{1}{sA_r} \left(n^m k\sqrt{g} (2h^m)^{-1/2} \right) < 0 \tag{26}$$

Moreover, after some computations:

$$\det(D_m) = \frac{\alpha P(w^m) k^2 g}{sA_r} > 0 \tag{27}$$

We can therefore conclude that our dynamical system is locally assymptotically stable.

(ii) <u>*Comparative statics at the steady state*</u>

Note that (texte)

$$\begin{aligned} h^m &= \tfrac{1}{2g} \left(\tfrac{c}{kP((1-\rho)R)} \right)^2, \, w^m = (1-\rho)R, \\ r^m &= k\sqrt{2gh^*} = \tfrac{c}{P((1-\rho)R)}, \\ n^m &= \frac{w^m}{r^m} = \frac{1}{c}(1-\rho)RP\left((1-\rho)R\right) = \tfrac{w^m P(w^m)}{c} \end{aligned}$$

From Eq.It follows that :

	∂c	∂k	∂R	$\partial \rho$
∂h^m	$\frac{c}{g}\left(\frac{1}{kP(w^m)}\right)^2 > 0$	$-\frac{1}{gk^3}\left(\frac{c}{P(w^m)}\right)^2 < 0$	$-\frac{\left(\frac{c}{k}\right)^2 P'(w^m)(1-\rho)}{g(P((w^m))^3} > 0$	$\frac{\left(\frac{c}{k}\right)^2 RP'(w^m)}{g(P(w^m))^3} < 0$
∂w^m	0	0	$(1-\rho) > 0$	$-R < 0$
∂r^m	$\frac{1}{P(w^m)} > 0$	0	$-\frac{c(1-\rho)P'(w^m)}{(P((1-\rho)R))^2} > 0$	$\frac{cRP'((1-\rho)R)}{(P(w^m))^2} < 0$
∂n^m	$-\frac{w^m P(w^m)}{c^2} < 0$	0	$\frac{(1-\rho)\left(P(w^m)+w^m P'(w^m)\right)}{c}$	$\frac{-R\left(P(w^m)+w^m P'(w^m)\right)}{c}$

Finally remark that $\frac{\partial n^m}{\partial R} \lesseqgtr 0 \Leftrightarrow \varepsilon_P(w^m) \lesseqgtr -1$ and $\frac{\partial n^m}{\partial R} \lesseqgtr 0 \Leftrightarrow \varepsilon_P(w^m) \gtreqless -1$

B. Proof of Proposition 3

(i) Existence and uniqueness of the steady state

By using condition (22), let us introduce the function:

$$\phi\left(h; A_r, R, \rho, k, c, \delta\right) \equiv P\left((1-\rho)\,R\right) - \frac{c}{k\sqrt{2gh}}\left(1 + \frac{(1-\rho)R}{2h\delta s A_r}\right) \tag{28}$$

which takes as parameter any $\theta \in \{A_r, R, \rho, k, c, \delta\}$. Let us first observe that:

- $\lim_{h\to 0}\phi\left(h\right) = \underbrace{P\left((1-\rho)\,R\right)}_{\text{finite}} - \underbrace{\left[\lim_{h\to 0}\frac{c}{k\sqrt{2gh}}\right]}_{=+\infty}\underbrace{\left(1 + \lim_{h\to 0}\frac{(1-\rho)R}{2h\delta s A_r}\right)}_{>1} = -\infty$
- $\lim_{h\to h^{\max}}\phi\left(h\right) = P\left((1-\rho)\,R\right) - \frac{c}{k\sqrt{2gh_{\max}}}\left(1 + \frac{(1-\rho)R}{2h\delta s A_r}\right) > P\left((1-\rho)\,R\right) - \frac{c}{k\sqrt{2gh_{\max}}} > 0$ since we have assumed $h_{\max}$ is high enough (see XXX)

This means that there exists at least on h^o which solves $\phi\left(h\right) = 0$. Moreover, since

$$\frac{\partial\phi}{\partial h} = \frac{c}{k\sqrt{g}\sqrt{(2h)^3}}\left(1 + \frac{3(1-\rho)R}{2h\delta s A_r}\right) > 0 \tag{29}$$

we can even conclude that this steady state is unique.

(ii) Local saddle point stability

In the following, the index t will be omitted to spear notation. Let us first remember that the dynamics is given by:

$$\left\{\begin{array}{l} \dot{\lambda} = \delta\lambda - \left.\partial_h\mathcal{H}\left(w,h,\lambda\right)\right|_{w=w(h,\lambda)} \\ \dot{h} = \left.\partial_\lambda\mathcal{H}\left(w,h,\lambda\right)\right|_{w=w(h,\lambda)} \end{array}\right. \quad \text{with } w(h,\lambda) \text{ solution to } \partial_w\mathcal{H}\left(w,h,\lambda\right) = 0 \tag{30}$$

It follows that the dynamics around the steady state can be approximated by:

$$\left(\begin{array}{c}\dot{\lambda}\\ \dot{h}\end{array}\right) = \underbrace{\left[\begin{array}{cc} \delta - \left(\partial^2_{h,\lambda}\mathcal{H} - \partial^2_{h,w}\mathcal{H}\frac{\partial^2_{w,\lambda}\mathcal{H}}{\partial^2_{w,w}\mathcal{H}}\right) & -\left(\partial^2_{h,h}\mathcal{H} - \partial^2_{h,w}\mathcal{H}\frac{\partial^2_{w,h}\mathcal{H}}{\partial^2_{w,w}\mathcal{H}}\right) \\ -\partial^2_{\lambda,w}\mathcal{H}\frac{\partial^2_{w,\lambda}\mathcal{H}}{\partial^2_{w,w}\mathcal{H}} & \partial^2_{\lambda,h}\mathcal{H} - \partial^2_{\lambda,w}\mathcal{H}\frac{\partial^2_{w,h}\mathcal{H}}{\partial^2_{w,w}\mathcal{H}} \end{array}\right]\Bigg|_{(w^o,h^o,\lambda^o)}}_{=D_o}\left(\begin{array}{c}\dot{\lambda} - \lambda^{opt}\\ \dot{h} - h^{opt}\end{array}\right) \tag{31}$$

with:

$$\left\{\begin{array}{lll} \partial^2_{h,\lambda}\mathcal{H} = 0 & \partial^2_{h,w}\mathcal{H} = \frac{c}{k\sqrt{g}}\left(2h\right)^{-\frac{3}{2}} & \partial^2_{h,h}\mathcal{H} = -\frac{3cw}{k\sqrt{g}}\left(2h\right)^{-\frac{5}{2}} \\ \partial^2_{\lambda,w}\mathcal{H} = -\frac{1}{sA_r} & \partial^2_{w,w}\mathcal{H} = P'(w) & \partial^2_{\lambda,\lambda}\mathcal{H} = 0 \end{array}\right. \tag{32}$$

By substitution:

$$D_o = \begin{bmatrix} \delta - \frac{c(2h)^{-\frac{3}{2}}}{sA_r k\sqrt{g}P'(w)} & \frac{3cw(2h)^{-\frac{5}{2}}}{k\sqrt{g}} + \left(\frac{c(2h)^{-\frac{3}{2}}}{k\sqrt{g}}\right)^2 \frac{1}{P'(w)} \\ -\left(\frac{1}{sA_r}\right)^2 \frac{1}{P'(w)} & \frac{c(2h)^{-\frac{3}{2}}}{sA_r k\sqrt{g}P'(w)} \end{bmatrix} \tag{33}$$

It follow that $trace(D_o) = \delta > 0$ and, after some simplifications, that

$$\det(D_o) = \left(\frac{c(2h)^{-\frac{3}{2}}}{sA_r k\sqrt{g}}\right)\left(\delta + \frac{3w}{2sA_r h}\right)\frac{1}{P'(w)} < 0 \tag{34}$$

Finally, since $trace(D_o) > 0$ and $\det(D_o) < 0$, we know that our steady state is a locally stable saddle point.

Sensitivity analysis at the optimal steady state values

We now compute the comparative statics on steady state value. let us remember our assumptions, we will use:

$P\left((1-\rho)R\right) - \frac{c}{k\sqrt{2gh}}\left(1 + \frac{(1-\rho)R}{2h\delta sA_r}\right)$

$$P'(w) < 0, : \rho(h) < 1, :: \rho'(h) > 0$$

Impact on the water table level

For all $\theta \in \{A_r, R, \rho, k, c, \delta\}$, let us study the sign of $\frac{\partial h}{\partial \theta} = -\frac{\partial \phi}{\partial \theta} \big/ \frac{\partial \phi}{\partial h}$ where ϕ is given by Eq. (23). First, we know from Eq.(29) that $\frac{\partial \phi}{\partial h} > 0$. We can say that $sign\{\frac{\partial h}{\partial \theta}\} = -sign\{\frac{\partial \phi}{\partial \theta}\}$. Let us compute $\frac{\partial \phi}{\partial \theta}$ for all $\theta \in \{A_r, R, \rho, k, c, \delta\}$:

$$\frac{\partial \phi}{\partial A_r} = \frac{c}{k\sqrt{2gh}}\left(\frac{(1-\rho)R}{2h\delta s(A_r)^2}\right) > 0, \quad \text{hence} \quad \frac{\partial h^{opt}}{\partial A_r} < 0 \tag{35}$$

$$\frac{\partial \phi}{\partial R} = (1-\rho)P'(w) - \frac{c}{k\sqrt{2gh}}\left(\frac{(1-\rho)}{2h\delta sA_r}\right) < 0, \quad \text{hence} \quad \frac{\partial h^{opt}}{\partial R} > 0 \tag{36}$$

$$\frac{\partial \phi}{\partial \rho} = -RP'(w) + \frac{c}{k\sqrt{2gh}}\left(\frac{R}{2h\delta sA_r}\right) > 0, \quad \text{hence} \quad \frac{\partial h^{opt}}{\partial \rho} < 0 \tag{37}$$

$$\frac{\partial \phi}{\partial k} = \frac{c}{k^2\sqrt{2gh}}\left(1 + \frac{(1-\rho)R}{2h\delta sA_r}\right) > 0, \quad \text{hence} \quad \frac{\partial h^{opt}}{\partial k} < 0 \tag{38}$$

$$\frac{\partial \phi}{\partial c} = -\frac{1}{k\sqrt{2gh}}\left(1 + \frac{(1-\rho)R}{2h\delta sA_r}\right) < 0, \quad \text{hence} \quad \frac{\partial h^{opt}}{\partial c} > 0 \tag{39}$$

$$\frac{\partial \phi}{\partial \delta} = \frac{c}{k\sqrt{2gh}}\left(\frac{(1-\rho)R}{2h\delta^2 sA_r}\right) > 0, \quad \text{hence} \quad \frac{\partial h^{opt}}{\partial \delta} < 0 \tag{40}$$

Impact on water consumption

At the steady state $w^{opt} = (1-\rho)R$, thus all $\theta \neq R, \rho$ $\frac{\partial w^{opt}}{\partial \theta} = 0$. Moreover, $\frac{\partial w^{opt}}{\partial R} = 1 - \rho > 0$ and $\frac{\partial w^{opt}}{\partial \rho} = -R < 0$

Impact on the yield of an artesian well

At the steady state $r(h^{opt}) = k\sqrt{2gh^{opt}}$, we observe, for all $\theta \neq k$, that $\frac{\partial r^{opt}}{\partial \theta} = \frac{kg}{\sqrt{2gh^{opt}}} \cdot \frac{\partial h^{opt}}{\partial \theta}$. It follows, respectively by equations Eq.(35), Eq.(37), Eq. (40) and by Eq.(36), Eq.(39), that $\frac{\partial r^{opt}}{\partial A_r}$, $\frac{\partial r^{opt}}{\partial \rho}$ $\frac{\partial r^{opt}}{\partial \delta} < 0$ and $\frac{\partial r^{opt}}{\partial R}$, $\frac{\partial r^{opt}}{\partial c} > 0$.

It remains to study $\frac{\partial r^{opt}}{\partial k} = \frac{kg}{\sqrt{2gh^{opt}}} \cdot \frac{\partial h^{opt}}{\partial k} + \sqrt{2gh^{opt}}$. Moreover $\frac{\partial h^{opt}}{\partial k} = -\frac{\partial \phi}{\partial k} \Big/ \frac{\partial \phi}{\partial h}$ and $\frac{\partial \phi}{\partial h} > 0$ (see Eq.(29)). We can therefore say that:

$$sign\left\{\frac{\partial r^{opt}}{\partial k}\right\} = sign\left\{k\frac{\partial h^{opt}}{\partial k} + 2h^{opt}\right\} = sign\left\{\underbrace{-k\frac{\partial \phi}{\partial k} + 2h^{opt}\frac{\partial \phi}{\partial h}}_{=D}\right\}$$

By using Eqs.(29) and (38), we obtain:

$$D = -\frac{c}{k\sqrt{2gh}}\left(1 + \frac{(1-\rho)R}{2h\delta sA_r}\right) + \frac{c}{k\sqrt{2gh}}\left(1 + \frac{3(1-\rho)R}{2h\delta sA_r}\right) = \frac{c}{k\sqrt{2gh}}\frac{2(1-\rho)R}{2h\delta sA_r} > 0$$

Impact on number of artesian wells

At the steady state $n^{opt} = \frac{(1-\rho)R}{k\sqrt{2gh^{opt}}}$, thus $\forall \theta \neq R, \rho, k$, $\frac{\partial n^{opt}}{\partial \theta} = -\frac{(1-\rho)R}{k\sqrt{g}}\left(2h^{opt}\right)^{-3/2}\frac{\partial h^{opt}}{\partial \theta}$ it follows, respectively by equations Eq.(35), Eq. (40) and Eq.(39), that $\frac{\partial n^{opt}}{\partial A_r}$, $\frac{\partial n^{opt}}{\partial \delta} > 0$ and $\frac{\partial n^{opt}}{\partial c} < 0$. Let us now concentrate on $\frac{\partial n^{opt}}{\partial k}$. By computation:

$$\frac{\partial n^{opt}}{\partial k} = -\frac{(1-\rho)R}{k^2\sqrt{2gh}} - \frac{(1-\rho)R}{k\sqrt{g}}\left(2h^{opt}\right)^{-3/2}\frac{\partial h^{opt}}{\partial k}$$

It follows that:

$$sign\left(\frac{\partial n^{opt}}{\partial k}\right) = -sign\left(\frac{2h}{k} + \frac{\partial h^{opt}}{\partial k}\right) = -sign\left(\frac{2h}{k}\frac{\partial \phi}{\partial h} - \frac{\partial \phi}{\partial k}\right)$$

Moreover by Eqs.(29) and (38)

$$\frac{2h}{k}\frac{\partial \phi}{\partial h} - \frac{\partial \phi}{\partial k} = \frac{c}{k^2\sqrt{2gh}}\left(1 + \frac{3(1-\rho)R}{2h\delta sA_r}\right) - \frac{c}{k^2\sqrt{2gh}}\left(1 + \frac{(1-\rho)R}{2h\delta sA_r}\right) = \frac{c}{k^2\sqrt{2gh}}\left(\frac{(1-\rho)R}{h\delta sA_r}\right) > 0$$

We can therefore say that $\frac{\partial n^{opt}}{\partial k} < 0$. Let us now move to the study of $\frac{\partial n^{opt}}{\partial \theta}$ for $\theta = R, \rho$. Since these two parameters always appear as $w^0 = (1-\rho)R$ we can say that:

$$\begin{aligned} sign\left(\frac{\partial n^{opt}}{\partial \theta}\right) &= sign\left(\frac{\partial n^{opt}}{\partial w^{opt}}\right) sign\left(\frac{\partial w^{opt}}{\partial \theta}\right) = sign\left(\frac{\partial w^{opt}}{\partial \theta}\right) sign\left(2h^{opt} - w^{opt}\frac{\partial h^{opt}}{\partial w^{opt}}\right) \\ &= sign\left(\frac{\partial w^{opt}}{\partial \theta}\right) sign\left(2h^{opt}\frac{\partial \phi}{\partial h} + w^{opt}\frac{\partial \phi}{\partial w^{opt}}\right) \end{aligned}$$

Moreover by Eqs.(28) and (29), we obtain:

$$\begin{aligned} \frac{1}{2h^{opt}}\frac{\partial \phi}{\partial h} + w^{opt}\frac{\partial \phi}{\partial w^{opt}} &= \frac{c}{k\sqrt{2gh^{opt}}}\left(1 + \frac{3w^{opt}}{2h^{opt}\delta sA_r}\right) + w^{opt}P'(w^{opt}) - \frac{c}{k\sqrt{2gh^{opt}}}\frac{w^{opt}}{2h^{opt}\delta sA_r} \\ &= \frac{c}{k\sqrt{2gh^{opt}}}\left(1 + \frac{w^{opt}}{h^{opt}\delta sA_r}\right) + w^{opt}P'(w^{opt}) \end{aligned}$$

By using again Eq.(28)

$$\begin{aligned} \frac{1}{2h^{opt}}\frac{\partial \phi}{\partial h} + w^{opt}\frac{\partial \phi}{\partial w^{opt}} &= P(w^{opt}) + \frac{c}{k\sqrt{2gh^{opt}}}\left(1 + \frac{w^{opt}}{2h^{opt}\delta sA_r}\right) + w^{opt}P'(w^{opt}) \\ &= P(w^{opt})\left(1 + \varepsilon_w(P)\right) + \frac{c}{k\sqrt{2gh^{opt}}}\left(1 + \frac{w^{opt}}{2h^{opt}\delta sA_r}\right) \end{aligned}$$

It follows that if the elasticity of the demand $\varepsilon_w(P) \geq -1$, we can assert that $\frac{1}{2h^{opt}}\frac{\partial \phi}{\partial h} + w^{opt}\frac{\partial \phi}{\partial w^{opt}} > 0$ which implies that $\frac{\partial n^{opt}}{\partial R} > 0$ and $\frac{\partial n^{opt}}{\partial R} < 0$.

This last condition is however only a sufficient condition. For $\varepsilon_w(P) < -1$, it is impossible to conclude. This can be easely illutrated by the following example. Let us take $P(w) = w^{-a}$ with $\alpha > 1$, and let us choose the parameters such that $(1-\rho)R = 1$ and $\frac{c}{k\sqrt{g}} = \frac{\delta s A}{\delta s A+1}$. Under these assumptions, we know from Eq.(28), that h^{opt} solves

$$1 - \tfrac{c}{k\sqrt{g}}(2h)^{-\frac{1}{2}} - \tfrac{c}{k\sqrt{g}}\tfrac{1}{\delta s A_r}(2h)^{-\frac{3}{2}} = 1 - \tfrac{\delta s A}{\delta s A+1}(2h)^{-\frac{1}{2}} - \tfrac{1}{\delta s A+1}(2h)^{-\frac{3}{2}} = 0 \qquad (41)$$

and it is easy to check that $h^{opt} = \frac{1}{2}$ solves Eq.(41). It follow that in this case

$$\tfrac{1}{2h^{opt}}\tfrac{\partial \phi}{\partial h} + w^{opt}\tfrac{\partial \phi}{\partial w^{opt}} = 1 - \alpha + \tfrac{c}{k\sqrt{g}}\left(1 + \tfrac{1}{\delta s A_r}\right) = 2 - \alpha$$

This means in this example that $\frac{\partial n^{opt}}{\partial R}$and $\frac{\partial n^{opt}}{\partial R}$ change their sign depending whenever the elasticity of the demand $\varepsilon_w(P) = -\alpha \lesseqgtr -2$

$$P\left((1-\rho)R\right) - \tfrac{c}{k\sqrt{2gh}}\left(1 + \tfrac{(1-\rho)R}{2h\delta s A_r}\right)$$

$$\begin{aligned}
\frac{\partial n^{opt}}{\partial R} &= \frac{(1-\rho)}{k\sqrt{2gh}}\left[\frac{c}{k\sqrt{g}\,(2h^{opt})^{3/2}}\left(1 + \frac{3(1-\rho)R}{2h\delta s A_r}\right)\right] + \frac{(1-\rho)\,R}{k\sqrt{g}\,(2h^{opt})^{3/2}}\left[(1-\rho)\,P'(w) - \frac{c(1-\rho)}{k\sqrt{2gh}2h\delta s A_r}\right] OK \\
&= \frac{(1-\rho)}{k\sqrt{g}\,(2h^{opt})^{3/2}}\left(P(w) + \frac{\text{deux}c(1-\rho)R}{k\sqrt{2gh}2h\delta s A_r}\right) + \frac{(1-\rho)\,R}{k\sqrt{g}\,(2h^{opt})^{3/2}}\left((1-\rho)\,P'(w) - \frac{c(1-\rho)}{k\sqrt{2gh}2h\delta s A_r}\right) \\
&= \frac{(1-\rho)}{k\sqrt{g}\,(2h^{opt})^{3/2}}\left(P(w) + (1-\rho)\,P'(w)R\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial n^{opt}}{\partial \rho} &= -\frac{\rho R}{k\sqrt{2gh}}\left[\frac{c}{k\sqrt{g}\,(2h^{opt})^{3/2}}\left(1 + \frac{3(1-\rho)R}{2h\delta s A_r}\right)\right] - \frac{(1-\rho)\,R}{k\sqrt{g}\,(2h^{opt})^{3/2}}\left[-P'(w) + \frac{cR}{k\sqrt{2gh}2h\delta s A_r}\right] \\
&= \frac{\rho R}{k\sqrt{2gh}}\left[P(w) + \frac{2c(1-\rho)R}{k\sqrt{2gh}2h\delta s A_r}\right] - \frac{(1-\rho)\,R}{k\sqrt{g}\,(2h^{opt})^{3/2}}\left[-P'(w) + \frac{cR}{k\sqrt{2gh}2h\delta s A_r}\right] \\
&= \frac{R}{k\sqrt{g}\,(2h^{opt})^{3/2}}\left[-\rho P(w) - (1-\rho)\,P'(w)\right] - \frac{R^2 c\,(1-\rho)\,(1+2\rho)}{k^2 g 4h^2\delta s A_r}
\end{aligned}$$

B.1. Study of the difference

Let ℓ be the long term proportion of water which is lost with respect to the optimal management (mal dit)

$$\ell\left(A_r, R, \rho, k, c, \delta\right) = 1 - \frac{h^m\left(A_r, R, \rho, k, c, \delta\right)}{h^o\left(R, \rho, k, c\right)}$$

where $h^m\left(A_r, R, \rho, k, c, \delta\right)$ and $h^o\left(R, \rho, k, c\right)$ respectively solve XXXX. If we denote by

$$x^o = (2h^o)^{-1/2} \text{ and } x^m = (2h^m)^{-1/2} = \sqrt{g}\left(\frac{kP\left((1-\rho)R\right)}{c}\right)$$

this quantity becomes $\ell = 1 - \left(\frac{x^o}{x^m}\right)^2$. Moreover if we set $w = (1-\rho)R$, $\kappa = \frac{c}{k\sqrt{g}}$ and $a = \delta s A_r$, we know from Eq. XX that x^o solves

$$P(w) - \kappa x - \tfrac{\kappa w}{a}x^3 = 0$$

This implies, since $x^m = \frac{P(w)}{\kappa}$, that $\lambda = \frac{x^o}{x^m}$ solves

$$P(w) - P(w)\lambda - \tfrac{w(P(w))^3}{a\kappa^2}\lambda^3 = 0$$
$$\Longleftrightarrow \lambda^3 + K(\lambda - 1) = 0 \text{ with } K = \tfrac{a\kappa^2}{w(P(w))^2} \text{ and } \lambda \in [0,1]$$

Moreover it is immediate by the implicit function theorem that

$$\frac{d\lambda}{dK} = -\frac{\lambda - 1}{3\lambda^2 + K} > 0$$

we can therefore assert that $\forall \theta \in \{A_r, R, \rho, k, c, \delta\}$,

$$\frac{\partial \ell}{\partial \theta} = -2\rho \frac{d\lambda}{dK}\frac{\partial K}{\partial \theta} \Rightarrow sign\left(\frac{\partial \ell}{\partial \theta}\right) = -sign\left(\frac{\partial K}{\partial \theta}\right)$$

Since $\frac{\partial K}{\partial a} = \frac{\kappa^2}{w(P(w))^2} > 0$ with $a = \delta s A_r$ we have that $\frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial A_r} < 0$. Moreover $\frac{\partial K}{\partial \kappa} = \frac{2a\kappa}{w(P(w))^2} > 0$ with $a = \frac{c}{k\sqrt{g}}$, it follows that $\frac{\partial r}{\partial c} < 0$ and $\frac{\partial r}{\partial k} > 0$. Finally note that

$$\frac{\partial K}{\partial w} = -\tfrac{a\kappa^2}{w^2(P(w))^3}\left(P(w) + 2wP'(w)\right) = -\tfrac{a\kappa^2 P(w)}{w^2(P(w))^3}\left(1 + 2e_P(w)\right)$$

and since $w = (1-\rho)R$, we can conclude that $\frac{\partial \ell}{\partial \rho} \gtreqless 0$ and $\frac{\partial \ell}{\partial R} \lesseqgtr 0$ if and only if $e_P(w) \lesseqgtr -\frac{1}{2}$.

Cadeau

If we denote by p proportion of yield that are lost par wells with respect to the steady state. We can say that:

$$p = 1 - \frac{r^m}{r^0} = 1 - \frac{k\sqrt{2gh^m}}{k\sqrt{2gh^0}} = 1 - \lambda$$

If follows that $\forall \theta \in \{A_r, R, \rho, k, c, \delta\}$, $sign\left(\frac{\partial \lambda}{\partial \theta}\right) = sign\left(\frac{\partial \ell}{\partial \theta}\right)$